# GEODESICALLY TRACKING QUASI-GEODESIC PATHS FOR COXETER GROUPS

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Abstract. If $\Lambda$ is the Cayley graph of a Gromov hyperbolic group, then it is a fundamental fact that quasi-geodesics in $\Lambda$ are tracked by geodesics. Let $(W, S)$ be a finitely generated Coxeter system and $\Lambda(W, S)$ the Cayley graph of $(W, S)$. For general Coxeter groups, not all quasi-geodesic rays in $\Lambda$ are tracked by geodesics. In this paper we classify the $\Lambda$-quasi-geodesic rays that are tracked by geodesics. As corollaries we show that if $W$ acts geometrically on a CAT(0) space $X$, then CAT(0) geodesics in $X$ are tracked by Cayley graph geodesics (where the Cayley graph is equivariantly placed in $X$) and for any $A \subset S$, the special subgroup $\langle A\rangle$ is quasi-convex in $X$. We also show that if $g$ is an element of infinite order for $(W, S)$ then the subgroup $\langle g\rangle$ is tracked by a Cayley geodesic in $\Lambda(W, S)$ (in analogy with the corresponding result for word hyperbolic groups).

## 1. Introduction

Suppose $G$ is a group with finite generating set $A$, and $\Lambda(G, A)$ is the Cayley graph of $G$ with respect to $A$. If $G$ is word hyperbolic then any quasi-geodesic in $\Lambda$ is tracked by a geodesic (see [Sh]). The corresponding result for CAT(0) groups is not true. Our main goal in this paper is to classify the quasi-geodesics in the Cayley graph of a finitely generated Coxeter system that are tracked by geodesics. We define a "bracket number" for a Cayley path in terms of the wall crossings of the path and our main theorem is that a quasi-geodesic ray or line is tracked by a geodesic iff the bracket number of the ray (line) is bounded. Our principal corollary to this theorem states that if $(W, S)$ is a finitely generated Coxeter system, and $W$ acts geometrically on a CAT(0) space $X$, then the CAT(0) geodesics of $X$ are tracked by $(W, S)$-Cayley geodesics in $X$. If $X$ is the Davis complex for $(W, S)$ or even if $W$ acts as a reflection group on $X$, the proof of the corollary is straightforward. Unfortunately, the reflection group argument has no analogue when $W$ does not act as a reflection group on $X$. The principal corollary directly implies that if $A \subset S$ then the special subgroup $\langle A\rangle$ is quasi-convex in $X$.

If a group $G$ acts geometrically on a CAT(0) space $X$ and one is interested in the asymptotic properties of $X$ it is a considerable advantage to know that CAT(0) geodesics in $X$ are tracked by Cayley geodesics. Clearly,

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the algebraic properties of $G$ are far more apparent in Cayley geodesics than in CAT(0) geodesics. This theme is highlighted in [MRT] where local connectivity of boundaries of right angled Coxeter groups are analyzed.

The work of B. Bowditch and G. Swarup (see [S]) imply that 1-ended word hyperbolic groups have locally connected boundary. One can easily see from our tracking results that any 1-ended hyperbolic Coxeter group has locally connected boundary.

## 2. Coxeter Preliminaries

We use M. Davis' book [D] as a general Coxeter group reference for this section. A Coxeter system is a pair $(W, S)$ where $S$ is a generating set for the group $W$ and $W$ has presentation

$$\langle S : (s_i s_j)^{m(i,j)} \text{ for all } s_i, s_j \in S\rangle$$

where $m(i, j) \in \{1, 2, \ldots, \infty\}$, $m(i, j) = 1$ iff $i = j$ (so all generators are order 2) and $m(i, j) = m(j, i)$. If $m(i, j) = \infty$, the element $s_i s_j$ is of infinite order (and the relation $(s_i s_j)^\infty$ is left out of the presentation).

A *reflection* in $W$ is a conjugate of an element of $S$. If $w \in W$ and $s \in S$ then the edge labeled $s$ in the Cayley graph $\Lambda(W, S)$ at the vertex $w$ is mapped to itself by the reflection $wsw^{-1}$, so that the vertices $w$ and $ws$ are interchanged. I.e. the edge is reflected across its midpoint. The set of edges in $\Lambda$ each fixed (set-wise) by a reflection is a *wall* of $\Lambda$. The walls of $\Lambda$ partition the edges of $\Lambda$ into disjoint sets. Notationally, we write a wall $Q$ as $[e]$ where $e$ is any edge of the wall $Q$ and we define $\bar{Q}$ to be the union of the edges of $Q$ in $\Lambda$. An edge $e$ (with say label $t \in S$) belongs to a wall $Q$ corresponding to the reflection $wsw^{-1}$ iff a vertex of $e$ is $wq$ where $qtq^{-1} = s$. The closure of the compliment of a wall in $\Lambda$ has exactly two components (which are interchanged by the reflection) called the *sides* of the wall. Two walls are *parallel* if all edges of one are on the same side of the other. If two walls are not parallel, then they *cross*. The following theorem due to B. Brink and R. Howlett (see theorem 2.8 of [BrH]) is a fundamental result concerning the wall structure of $\Lambda$.

**Theorem 2.1. (Parallel Wall theorem)** *Suppose $(W, S)$ is a finitely generated Coxeter system and $\Lambda(W, S)$ the Cayley graph of $W$ with respect to $S$. For each positive integer $n$ there is a constant $P(n)$ such that the following holds: given a wall $Q$ and a point $p$ in $\Lambda$ such that the distance from $p$ to $\bar{Q}$ is at least $P(n)$, then there exist $n$ distinct pairwise parallel walls which separate $\bar{Q}$ from $p$.*

For a path $\beta$ in $\Lambda$ and vertex $t$ of $\beta$ let the *bracket number of $t$ in $\beta$* be the number of walls $Q$ such that there is an edge of $Q$ on either side of $t$ in $\beta$. Denote the bracket number of $t$ in $\beta$ as $B(t, \beta)$. If $\tau$ is a subpath of $\beta$ the *bracket number of $\tau$ in $\beta$* is the maximum of the numbers $B(t, \beta)$ for all vertices $t$ of $\tau$. Denote this number $B(\tau, \beta)$. Call $B(\beta) \equiv B(\beta, \beta)$ the *bracket number of $\beta$*.

## 3. Wall computations

If $\alpha$ is an edge path in the Cayley graph $\Lambda$ with consecutive vertices $a = v_0, v_1, \ldots, v_n = b$, then an *$L$-approximation to* $\alpha$ is an edge path in $\Lambda$ connecting $a$ and $b$ of the form $(\alpha_1, \ldots, \alpha_n)$ where for all $i$, $\alpha_i$ is geodesic connecting $w_{i-1}$ to $w_i$ and $w_i$ is within $L$ of $v_i$. The points $w_i$ are called *approximation points.*

**Lemma 3.1.** *Suppose $(W, S)$ is a finitely generated Coxeter system, $\alpha$ is an edge path in the Cayley graph $\Lambda(W, S)$ connecting $a$ and $b$, and $\beta$ is an $L$-approximation of $\alpha$. Then the bracket number $B(\beta)$ is bounded by a constant only depending on $B(\alpha)$, $L$ and constants independent of the choice of $\alpha$.*

*Proof.* Let the consecutive vertices of $\alpha$ be $a = v_0, v_1, \ldots, v_n = b$, the approximation vertices of $\beta$ be $a = w_0, w_1, \ldots, w_m = b$ (so that $d_\Lambda(w_i, v_i) \leq L$ for all $i$) and $\beta_i$ be the geodesic subpath of $\beta$ connecting $w_{i-1}$ to $w_i$. Then $\beta = (\beta_1, \ldots, \beta_m)$. If $x$ is a vertex of $\beta_i$ and $B(x, \beta)$ is "large", then (as each edge belongs to exactly one wall) there is a wall $Q$ that brackets $x$ on $\beta$ that is "far" from $x$ and hence far from $v_i$. Hence it suffices to bound the distance between $v_i$ and a wall $Q$ that brackets $x$ on $\beta$. The Parallel Wall theorem implies this distance is large iff there is a large set $\mathcal{Q}$ of (mutually parallel) walls that separate $\bar{Q}$ from $v_i$, so it suffices to bound the size of the set $\mathcal{Q}$ of walls that separate $\bar{Q}$ from $x$. Say $j < i < k$ such that $e_j$ and $e_k$ are edges of $\beta_j$ and $\beta_k$ respectively, and each of $e_j$, $e_k$ belongs to the wall $Q$. (See figure 1)

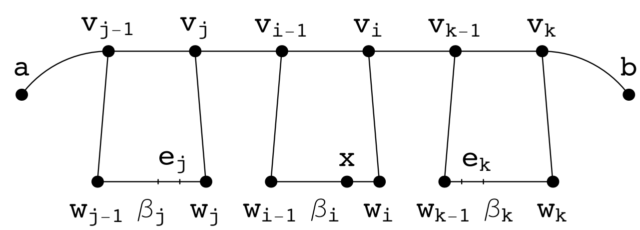


Figure 1

A path $\delta_j$, that begins at the end point of $e_j$ follows $\beta_j$ to $w_j$ and then travels geodesically from $w_j$ to $v_j$ has length $\leq 3L$. If $\alpha_{j,i}$ is the subpath of $\alpha$ from $v_j$ to $v_i$, then the path $(\delta_j, \alpha_{j,i})$ must cross each wall of $\mathcal{Q}$. Similarly define a path from $e_k$ to $v_i$ (which also crosses each wall of $\mathcal{Q}$). Then at most $6L$ walls of $\mathcal{Q}$ do not bracket $v_i$ on $\alpha$. This bounds the size of $\mathcal{Q}$ by $6L + B(\alpha)$. $\square$

**Lemma 3.2.** *Suppose $(W, S)$ is a Coxeter system and $\alpha = (e_1, \ldots, e_n)$ is a geodesic edge path connecting vertices $a$ and $b$ in $\Lambda(W, S)$ such that $\alpha$ does not cross the wall $Q$. If $e_0$ is an edge at $a$ and $e_{n+1}$ an edge at $b$ such that $e_0$ and $e_{n+1}$ belong to the wall $Q$ then each vertex of $\alpha$ is within $P(1)$ of $\bar{Q}$ (where $P$ is the function of theorem 2.1). In particular, if $v$ is a vertex of $\alpha$ and $v'$ the reflection of $v$ across $Q$ then $d(v, v') \leq 2P(1) + 1$.*

*Proof.* Otherwise, there is a wall $Q'$ separating a vertex $v$ of $\alpha$ from $Q$. Hence there is an edge of $\alpha$ between $a$ and $v$ that belongs to $Q'$ and an edge of $\alpha$ between $v$ and $b$ that belongs to $Q'$. This is impossible as $\alpha$ is geodesic. □

**Proposition 3.3.** *Suppose $(W, S)$ is a Coxeter system and $\alpha$ is an edge path of $\Lambda(W, S)$ connecting $a$ and $b$. Then there is an $L$-approximation $\beta$ to $\alpha$ such that each vertex of $\beta$ is on a geodesic connecting $a$ and $b$ and such that $L \leq (2P(1) + 1)B(\alpha)$.*

*Proof.* Let the consecutive vertices of $\alpha$ be $a = v_0, \ldots, v_n = b$. For $0 < i < n$ we choose an approximation point $w_i$ for $v_i$ as follows. Let $\alpha_i$ be the geodesic from $a$ to $v_i$ and $\beta_i$ the geodesic from $v_i$ to $b$. Each wall of $(\alpha_i, \beta_i)$ is crossed exactly once or twice. The number of walls crossed twice by $(\alpha_i, \beta_i)$ is

$$N_i \equiv \frac{1}{2}(d(a, v_i) + d(v_i, b) - d(a, b)) \leq B(\alpha)$$

Let $e$ be the last edge of $\alpha_i$ belonging to a wall which is crossed twice by $(\alpha_i, \beta_i)$ and $d$ the edge of $\beta_i$ in the same wall as $e$. (See figure 2.)

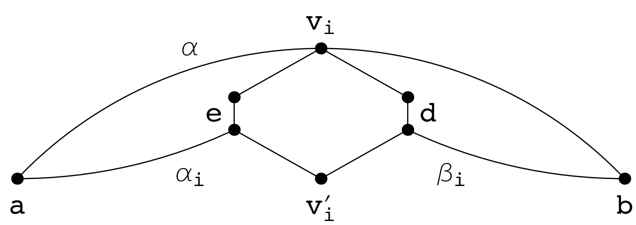


Figure 2

The segment of $(\alpha_i, \beta_i)$ between $e$ and $d$ is geodesic. Considering the reflection of this segment across the wall containing $e$ and $d$ (equivalently, delete $e$ and $d$ from $(\alpha_i, \beta_i)$). Then we see that $v_i'$, the reflection of $v_i$, is within $2P(1)+1$ of $v_i$ (lemma 3.2), and the distance from $v_i'$ to $a$ (respectively $b$) is less than that of $v_i$ to $a$ (respectively $b$). Hence $\frac{1}{2}(d(a, v_i') + d(v_i', b) - d(a, b)) < N_i$ and a geodesic from $a$ to $v_i'$ followed by a geodesic from $v_i'$ to $b$ crosses at most $N_i - 1$ walls twice. Continuing as above at most $N_i (\leq B(\alpha))$ such reflections are needed to move $v_i$ to a point $w_i$ on a geodesic between $a$ and $b$, and so $d(w_i, v_i) \leq (2P(1) + 1)B(\alpha)$.

It remains to see that each vertex of a geodesic connecting $w_i$ and $w_{i+1}$ belongs to a geodesic connecting $a$ and $b$. Consider the edge path $(\delta_i, \beta_i, \gamma_i)$ where $\delta_i$ is a geodesic connecting $a$ to $w_i$, $\beta_i$ is a geodesic connecting $w_i$ to $w_{i+1}$ and $\gamma_i$ is a geodesic connecting $w_{i+1}$ to $b$. The paths $\delta_i$ and $\gamma_i$ only cross walls crossed by some (equivalently any) geodesic connecting $a$ to $b$. If a vertex $v$ of $\beta_i$ is not on a geodesic connecting $a$ and $b$ then there is a wall $R$ separating $v$ from some (equivalently every) geodesic connecting $a$ and $b$. As $R$ separates $v$ from $a$, and $\delta_i$ does not cross $R$, $\beta_i$ must cross $R$ between

$w_i$ and $v$. Similarly $\beta_i$ must cross $R$ between $v$ and $w_{i+1}$. This is impossible as $\beta_i$ is geodesic. □

If $\gamma$ is an edge path in $\Lambda$ connecting the vertices $a$ and $b$, then each wall separating $a$ and $b$ is crossed an odd number of times by $\gamma$ and each wall not separating $a$ and $b$ is crossed and even number of times by $\gamma$. If two edges of $\gamma$ belong to the same wall then they may be "deleted" to obtain another path from $a$ to $b$ (i.e. if edges $e$ and $d$ of $\gamma$ belong to the same wall $Q$, and $\tau$ is the segment of $\gamma$ between $e$ and $d$, then $(e, \tau, d)$ can be replaced in $\gamma$ by $\tau'$, where $\tau'$ is the reflection of $\tau$ across $Q$, to obtain a shorter path connecting $a$ and $b$). If $\alpha$ is a geodesic connecting $a$ and $b$ then the walls separating $a$ and $b$ are the walls determined by the edges of $\alpha$, so the walls separating $a$ and $b$ are in 1-1 correspondence with the edges of some (any) geodesic connecting $a$ and $b$. The following observations are straightforward.

**Lemma 3.4.** *Suppose $\beta$ is an edge path in $\Lambda$ connecting the vertices $a$ and $b$ such that each vertex of $\beta$ is on a geodesic connecting $a$ and $b$. Then*

*i) each edge of $\beta$ belongs to a wall that separates $a$ from $b$,*

*ii) each wall crossed by $\beta$ is crossed an odd number of times, and*

*iii) if $c$ and $d$ are vertices of $\beta$ then any wall separating $c$ and $d$ also separates $a$ and $b$.*

The next result is a slightly more sophisticated version of lemma 3.2.

**Lemma 3.5.** *Suppose $\alpha$ is a geodesic edge path in $\Lambda$ connecting the vertices $a$ and $b$, $v$ is a vertex of $\alpha$, and $a$ and $b$ are each within distance $A$ of $\bar{Q}$ for some wall $Q$. Then $v$ is within distance $2A(2P(1)+1)+P(1)$ of $\bar{Q}$.*

*Proof.* Let $a'$ (respectively $b'$) be a vertex of $\bar{Q}$ within $A$ of $a$ (respectively $b$) and on the same side of $Q$ as is $a$ (respectively $b$). Let $\beta$ (respectively $\gamma$) be a geodesic from $a'$ to $a$ (respectively $b$ to $b'$).

**Case 1.** The geodesic $\alpha$ does not cross $Q$.
In this case the path $\delta_0 \equiv (\beta, \alpha, \gamma)$ does not cross $Q$. Since $|\beta| \leq A$ and $|\gamma| \leq A$, a sequence of at most $2A$ deletions (the first in the path $\delta_0$) determines a geodesic connecting $a'$ to $b'$ which does not cross $Q$.

$*$) Each deletion is taken so that if $e$ and $d$ are the deleted edges, then the subpath determined by $e$ (or $d$) along with the subpath between $e$ and $d$ is geodesic.

If $e_1$ and $d_1$ are the first such deletion edges (so $e_1$ and $d_1$ are edges of $\delta_0$) then let $\delta_1$ be obtained from $\delta_0$ by deleting $e_1$ and $d_1$. If $v$ is not between $e_1$ and $d_1$ then $v$ is a vertex of $\delta_0$. If $v$ is between $e_1$ and $d_1$, then $v_1$, the reflection of $v$ across the wall $[e_1] = [d_1]$, is within $2P(1)+1$ of $v$, by lemma 3.2. (Note that the hypotheses of lemma 3.2 are satisfied since we require condition $*$.) In any case $\delta_1$ contains a vertex $v_1$ within $2P(1)+1$ of $v$. If $e_2$ and $d_2$ are deleting edges of $\delta_1$ (satisfying $*$), then let $\delta_2$ by obtained from $\delta_1$ by deleting $e_2$ and $d_2$. Lemma 3.2 implies $\delta_2$ contains a vertex $v_2$ within

$2P(1)+1$ of $v_1$ and so within $2(2P(1)+1)$ of $v$. Inductively, after $K \leq 2A$ deletions, we obtain a geodesic $\delta_K$ connecting $a'$ and $b'$, and $\delta_K$ contains a vertex $v_K$ within $K(2P(1)+1)$ of $v$. Note that $\delta_k$ does not cross $Q$. By lemma 3.2, $v_K$ is within $P(1)$ of $\bar{Q}$ so that $v$ is within $2A(2P(1)+1)+P(1)$ of $\bar{Q}$. This completes case 1.

**Case 2.** Suppose $\alpha$ crosses $Q$.
Say the edge $e$ of $\alpha$ between $v$ and $b$ belongs to $Q$. Repeat the case 1 argument with $\delta_0$ replaced by $(\beta, \alpha')$, where $\alpha'$ is the subsegment of $\alpha$ from $a$ to the initial point of $e$. Similarly if $e \in Q$ is an edge of $\alpha$ between $a$ and $v$. Note that in both case 2 scenarios, at most $A$ deletions are required to straighten to a geodesic, so the bound is reduced to $A(2(P(1)+1)+P(1)$. $\square$

## 4. Tracking Quasi-geodesics

We are interested in quasi-geodesic edge paths in $\Lambda$. An *edge path* in $\Lambda$ is a continuous map $\beta : [0, n] \to \Lambda$ such that $n \in \mathbb{Z}^+$ and for each non-negative integer $k$, $\beta$ maps the interval $[k, k+1]$ isometrically to an edge of $\Lambda$. Similarly if $\beta : [0, \infty) \to \Lambda$, then $\beta$ is called a *ray* and, if $\beta : (-\infty, \infty) \to \Lambda$ then $\beta$ is called a *line*. An edge path $\beta$ is a $(\lambda, \epsilon)$-*quasi-geodesic* if for each pair of integers $s$ and $t$, $|s-t| \leq \lambda d(\beta(s), \beta(t)) + \epsilon$. If $\alpha$ and $\beta$ are edge paths, then $\beta$ is $K$-*tracked* by $\alpha$ if each vertex of $\beta$ is within $K$ of a vertex of $\alpha$.

**Lemma 4.1.** *For $i \in \{1, 2\}$ suppose $\beta_i$ is a $(\lambda_i, \epsilon_i)$-quasi-geodesic edge path in $\Lambda$, $\beta_1$ is $K$-tracked by $\beta_2$ and $\beta_1(0)$ is within $K$ of $\beta_2(0)$. Assume both $\beta_1$ and $\beta_2$ are bi-infinite, or both are rays, or both are finite length and the terminal points of $\beta_1$ and $\beta_2$ are within $K$ of one another. Then $\beta_2$ is $(\lambda_2(2K+1)+\epsilon_2+K)$-tracked by $\beta_1$.*

*Proof.* Since each vertex of $\beta_1$ is within $K$ of a vertex of $\beta_2$, we may define an integer function $a$ such that for each integer $i$ (in the domain of $\beta_1$), $\beta_1(i)$ is within $K$ of $\beta_2(a(i))$. We take $a(0) = 0$ and if $\beta_i$ has $n_i$ edges then $a(n_1) = n_2$.

The first two inequalities follow from the definitions and the third follows from the first two.

**1)** $$\frac{|a(m+i)-a(m)|-\epsilon_2}{\lambda_2} - 2K \leq d(\beta_2(a(m+i)), \beta_2(a(m))) - 2K \leq$$

$$d(\beta_1(m+i), \beta_1(m)) \leq d(\beta_2(a(m+i)), \beta_2(a(m)))+2K \leq |a(m+i)-a(m)|+2K$$

**2)** $$\frac{i-\epsilon_1}{\lambda_1} \leq d(\beta_1(m+i), \beta_1(m)) \leq i$$

**3)** $$\frac{i-\epsilon_1}{\lambda_1} - 2K \leq |a(m+i)-a(m)| \leq$$

$$\lambda_2(d(\beta_1(m+i),\beta_1(m))+2K)+\epsilon_2 \leq (i+2K)\lambda_2+\epsilon_2$$

The inequality $|a(i+1)-a(i)| \leq \lambda_2(2K+1)+\epsilon_2$ implies if $k$ is between $a(i)$ and $a(i+1)$ for some $i$ then $\beta_2(k)$ is within $\lambda_2(2K+1)+\epsilon_2+K$ of $\beta_1(i)$. In the case $\beta_1$ and $\beta_2$ are finite, the condition that terminal points are within $K$ of one another (so that $a(n_1)=n_2$) implies that every integer in the domain of $\beta_2$ is between $a(i)$ and $a(i+1)$ for some $i$ and this case is finished. If $\beta_1$ and $\beta_2$ are rays then $a(i)$ is non-negative and equation 3) (with $m=0$) implies $a(i)$ is arbitrarily large for large $i$ and again every integer in the domain of $\beta_2$ is between $a(i)$ and $a(i+1)$ for some $i$. If $\beta_1$ and $\beta_2$ are bi-infinite, then the $a(i)$ may be positive or negative and (again by 3)) for large $|i|$, $|a_i|$ is large, and $lim_{i\to+\infty}a(i)=\pm\infty$ and $lim_{i\to-\infty}a(i)=\pm\infty$. It remains to see $lim_{i\to+\infty}a(i)\neq lim_{i\to-\infty}a(i)$. Equality is impossible, since otherwise, for every large positive integer $i$, $a(-i)$ would be between $a(j)$ and $a(j+1)$ for some (depending on $i$) large positive integer $j$. But equation 3) implies $a(j)$ and $a(j+1)$ are relatively close and $a(-i)$ and $a(j)$ are far apart. □

**Proposition 4.2.** *Suppose $\beta$ is a quasi-geodesic edge path ray in $\Lambda$ and $\beta$ is tracked by a geodesic, then $\beta$ has bounded bracket number.*

*Proof.* Assume that $\beta$ is a $(\lambda,\epsilon)$-quasi-geodesic. Suppose $\alpha$ is a geodesic such that each vertex of $\beta$ is within $L$ of a vertex of $\alpha$. For each integer $n\geq 0$, choose an integer $a(n)$ such that $d(\beta(n),\alpha(a(n)))\leq L$. We assume that $a(0)=0$.

The next two equations follow from the definitions and the third follows from the first two.

$$a(n)-2L\leq d(\beta(n),\beta(0))\leq a(n)+2L$$

$$\frac{n-\epsilon}{\lambda}\leq d(\beta(n),\beta(0))\leq n$$

$$\frac{n-\epsilon}{\lambda}-2L\leq a(n)\leq n+2L$$

**Claim 4.3.** *Suppose $K$ is an integer larger than $\lambda(4L+1)+\epsilon$. Then for any integer $n$, $a(n+K)>a(n)$.*

*Proof.* Note that if $m\geq\lambda(n+4L)+\epsilon$ then $a(m)>n+2L>a(n)$. So if $K>\lambda(4L+1)+\epsilon$, and $a(n+K)\leq a(n)$, then there is a last integer $K_1>\lambda(4L+1)+\epsilon$ such that $a(n+K_1)\leq a(n)$. Then (see figure 3)

$$a(n+K_1+1)>a(n)\geq a(n+K_1)$$

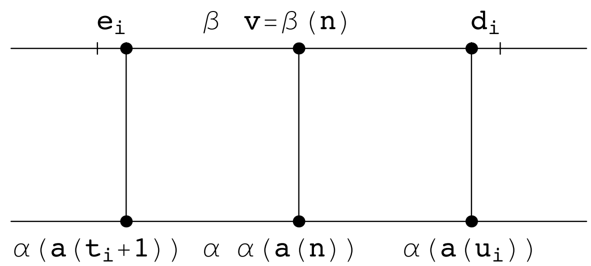


Figure 3

Since $d(\beta(n+K_1), \beta(n+K_1+1)) = 1$ for all $n$, and $d(\beta(i), \alpha(a(i))) \leq L$ for all $i$, we have

$$d(\alpha(a(n+K_1)), \alpha(a(n+K_1+1))) \leq 2L+1$$

But as $\alpha(a(n))$ is between $\alpha(a(n+K_1))$ and $\alpha(a(n+K_1+1))$ on the geodesic $\alpha$,

$$d(\alpha(a(n)), \alpha(a(n+K_1+1))) \leq 2L+1$$

Then $d(\beta(n), \beta(n+K_1+1)) \leq 4L+1$. But

$$d(\beta(n), \beta(n+K_1+1)) \geq \frac{1}{\lambda}(K_1+1-\epsilon) > 4L+1$$

the desired contradiction (so the claim is proved). □

Now suppose $v \equiv \beta(n)$ is a vertex of $\beta$ with bracket number at least $2\lambda(4L+1)+2\epsilon+K$. Then (by the pigeon hole principal) there are $K$ distinct walls, $Q_1, \ldots, Q_K$ such that for each $i \in \{1, \ldots, K\}$, there is an edge $e_i$ of $\beta$ preceding $v$ and an edge $d_i$ of $\beta$ following $v$ such that $e_i$ and $d_i$ belong to the wall $Q_i$, the subpath of $\beta$ between $e_i$ and $d_i$ does not cross $Q_i$, $e_i$ is not one of the $\lambda(4L+1)+\epsilon$ edges of $\beta$ immediately preceding $v$ and $d_i$ is not one of the $\lambda(4L+1)+\epsilon$ edges of $\beta$ immediately following $v$. I.e. $e_i = \beta([t_i, t_i+1])$ where $t_i + 1 \leq n - \lambda(4L+1) - \epsilon$ and $d_i = \beta([u_i, u_i+1])$ where $u_i \geq n + \lambda(4L+1) + \epsilon$. (See figure 4.)

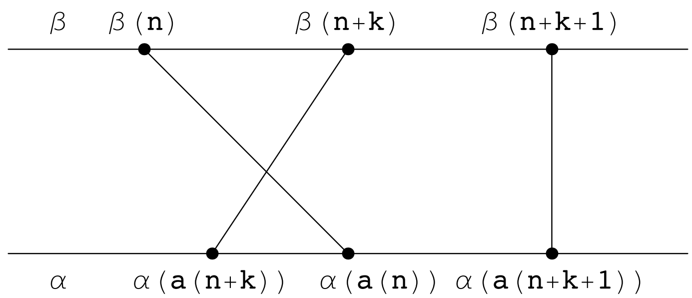


Figure 4

By claim 4.3, $a(t_i + 1) < a(n) < a(u_i)$. Hence, by lemma 3.5, $\alpha(a(n))$ is within $2L(2P(1) + 1) + P(1)$ of the wall $Q_i$. For $x$ a vertex of $\Lambda$, let $C(k)$ be the number of distinct walls that pass within $k$ of $x$. Note that $C$ is independent of vertex in $\Lambda$. Hence $K \leq C(2L(2P(1)+1)+P(1))$, bounding the bracket number of a vertex of $\beta$. $\square$

## 5. Proof of Main Theorem

In order to prove the main theorem, we need two results, one due to B. Brink and R. Howlett [BrH], and a second, due to R. P. Dilworth [Di].

**Theorem 5.1.** *(Brink-Howlett) Suppose $(W, S)$ is a finitely generated Coxeter system, and $\Lambda(W, S)$ is the Cayley graph of $W$ with respect to $S$. There is a bound $F_{(W,S)}$ on the number of mutually crossing walls of $\Lambda$.*

Dilworth's theorem requires several definitions. If $A$ is a partially ordered set (a set with reflexive, antisymmetric and transitive binary relation $\leq$ on $A$), then any two elements $x$ and $y$ are *comparable* if either $x \leq y$ or $y \leq x$. Otherwise they are in *incomparable*. A subset $C$ of $A$ is a *chain* when every pair of points in $C$ is a comparable pair. A subset $B$ of $A$ is called an *anitchain* when every pair of points in $B$ is an incomparable pair. The number of points in a maximal antichain is called the *width* of $A$.

**Theorem 5.2.** *(Dilworth) If $A$ is a partially ordered set of width $w$, then $A$ can be partitioned into $w$ chains.*

Suppose $x$ and $y$ are vertices of $\Lambda(W, S)$ and $\mathcal{W}_{(x,y)}$ is the set of walls that separate $x$ and $y$. We partially order $\mathcal{W}_{(x,y)}$ by saying $P \leq Q$ if either $P = Q$, or $P$ and $Q$ are parallel and $P$ separates $x$ from $Q$. Note that $P$ and $Q$ are parallel walls of $\mathcal{W}_{(x,y)}$, iff they are comparable. Hence $P$ and $Q$ are incomparable iff they cross. By proposition 5.1, the width of $\mathcal{W}_{(x,y)}$ is $F_{(W,S)}$. Applying Dilworth's theorem we have:

**Proposition 5.3.** *Suppose $(W, S)$ is a finitely generated Coxeter system, and $\Lambda(W, S)$ is the Cayley graph of $W$ with respect to $S$. For any vertices $x$ and $y$ of $\Lambda$ the walls separating $x$ and $y$ can be partitioned into at most $F_{(W,S)}$ chains (where any two walls in the same chain are parallel).*

Say a path is *geodesic with respect to a set of walls* if the path crosses each wall of the set either 0 or 1 times. The following lemma is clear.

**Lemma 5.4.** *Suppose $\alpha$ is an edge path in $\Lambda$ and $\alpha$ is geodesic with respect to the set of parallel walls $\mathcal{Q}$. If a subpath of $\alpha$ is replaced by a geodesic edge path, then the resulting edge path is geodesic with respect to $\mathcal{Q}$.*

**Theorem 5.5.** *Suppose $(W, S)$ is a finitely generated Coxeter system, $\alpha$ is a $(\lambda, \epsilon)$-quasi-geodesic edge path from $a$ to $b$ in the Cayley graph $\Lambda(W, S)$. Then there is an integer $K$, depending only on $\Lambda$, $\lambda$, $\epsilon$ and the bracket number $B$ of $\alpha$, and a $\Lambda$-geodesic $\beta$ connecting $a$ and $b$ such that $\alpha$ is $K$-tracked by $\beta$.*

*Proof.* The proof is a double induction argument. By proposition 5.3, the walls separating $a$ and $b$ can be partitioned into at most $F$ sets $\mathcal{Q}_1, \ldots, \mathcal{Q}_A$, where two walls in the same set are parallel. The "outside" induction is on the number $A(\leq F)$ of sets of walls separating $a$ and $b$. The fact that $A$ is bounded by $F$ is critical to the argument that follows. Note that if $A = 1$ then all walls separating $a$ and $b$ are parallel. In this case, the walls separating $a$ and $b$ are ordered as $Q_1, \ldots, Q_m$ where for $i < j < k$, $Q_j$ separates $Q_i$ from $Q_k$. Hence, there is a unique, geodesic edge path $\beta$ connecting $a$ and $b$, and $\beta$ crosses $Q_1$, then $Q_2$, .... By proposition 3.3, the path $\alpha$ is approximated by a path $\alpha'$, such that each vertex of $\alpha'$ is on a geodesic connecting $a$ and $b$. The path $\alpha'$ only crosses the walls separating $a$ and $b$ (see lemma 3.4) and, in this case, is geodesic, modulo backtracking. Eliminating backtracking on $\alpha'$ produces $\beta$. Each vertex of $\alpha'$ is a vertex of $\beta$ and the basis case is complete.

Assume the statement of the theorem is true if $A$, the number of sets of walls separating $a$ and $b$, is less than or equal to $M - 1$. Suppose there are $M$ sets of walls $(\mathcal{Q}_1, \ldots, \mathcal{Q}_M)$ separating $a$ and $b$. By proposition 3.3 we may assume every vertex of $\alpha$ is on a geodesic connecting $a$ and $b$, so that $\alpha$ only crosses walls separating $a$ and $b$ and $\alpha$ crosses each such wall an odd number of times. The second induction is on $N(\leq M)$, the number of sets of walls, $\mathcal{Q}_i$, such that $\alpha$ is not geodesic with respect to $\mathcal{Q}_i$. If $N = 0$, then $\alpha$ is geodesic. Assume the statement of the theorem is true for $N = K - 1$ (when the number of sets of walls separating $a$ and $b$ is $\leq M$). Assume the $\mathcal{Q}_i$ are arranged so that $\alpha$ is geodesic with respect to $\mathcal{Q}_i$ for $K + 1 \leq i \leq M$. Write $\alpha$ as $(e_1, \ldots, e_n)$ with consecutive vertices $a \equiv a_1, \ldots, a_n \equiv b$. Let $i$ be the first integer such that $e_i$ is an edge of a wall of $\mathcal{Q}_K$ and for some $j > i$, $e_j$ and $e_i$ are in the same wall $Q$. Now assume $j$ is the largest integer such that $e_j \in Q$. Since $\alpha$ crosses $Q$ an odd number of times, the path $\alpha_{i,j} \equiv (e_i, \ldots, e_{j-1})$ (from $a_i$ to $a_j$) crosses $Q$ an even number of times. A geodesic $\beta_{i,j}$ connecting $a_i$ to $a_j$ does not cross $Q$. Since all walls of $\mathcal{Q}_K$ are

parallel to one another, $\beta_{i,j}$ does not cross a wall of $\mathcal{Q}_K$. Hence $a_i$ and $a_j$ are not separated by a wall of $\mathcal{Q}_K$. By proposition 3.3, $\alpha_{i,j}$ is close to $\alpha'_{i,j}$ a quasi-geodesic edge path connecting $a_i$ to $a_j$, such that each vertex of $\alpha'_{i,j}$ is on a geodesic connecting $a_i$ to $a_j$. By lemma 3.4, each wall separating $a_i$ and $a_j$ also separates $a$ and $b$, and the number of sets of walls separating $a_i$ and $a_j$ is less than $M$. By (outside) induction, there is a geodesic $\beta_{i,j}$ connecting $a_i$ and $a_j$ which tracks $\alpha'_{i,j}$ and therefore tracks $\alpha_{i,j}$. Replace $\alpha_{i,j}$ by $\beta_{i,j}$. The resulting path, $\alpha_1$ crosses $Q$ exactly once at $e_j$. The walls of $\mathcal{Q}_K$ are ordered as $Q_1, Q_2, \ldots$ so that if $i < j$, then $Q_i$ separates $a$ from $Q_j$, and $Q_j$ separates $Q_i$ from $b$. A wall of $\mathcal{Q}_K$ preceding $Q$ in this ordering is not crossed by $\alpha_1$ after $e_j$. Hence if $\mathcal{Q} \subset \mathcal{Q}_K$ is the set of walls of $\mathcal{Q}_K$ preceding $Q$ and including $Q$, then $\alpha_1$ is geodesic with respect to $\mathcal{Q}$ and (by lemma 5.4), $\alpha_1$ is geodesic with respect to each set $\mathcal{Q}_i$ for $i > K$. Suppose $e_k$ is the first edge of $\alpha_1$ such that $e_k$ is an edge of a wall $Q$ of $\mathcal{Q}_K$, and for some $l > k$, $e_l \in Q$. Then $e_k$ follows $e_j$ on $\alpha_1$, and if we assume $e_l$ is the last edge of $\alpha_1$ in $Q$, then as above $(e_k, \ldots, e_{l-1})$ can be replaced by a geodesic close to $(e_k, \ldots, e_{l-1})$. Continuing, the resulting path is geodesic with respect to $\mathcal{Q}_K$ and by induction, the theorem follows. Note that the bound $F$ for $\Lambda$ (on the number of sets of parallel walls are necessary to partition the set of walls separating two points $a$ and $b$ of $\Lambda$), limits the total number of times the induction steps are carried out to arrive at a geodesic. $\square$

## 6. Consequences of the Main Theorem

**Corollary 6.1.** *Suppose $(W, S)$ is a finitely generated Coxeter system, and $\Lambda(W, S)$ is the Cayley graph of $W$ with respect to $S$. Any infinite or bi-infinite $(\lambda, \epsilon)$-quasi-geodesic edge path $\alpha$ with bounded bracket number $B$ is $K'$-tracked by an edge path geodesic where $K'$ is a constant only depending on $\lambda$, $\epsilon$, $B$ and $S$.*

*Proof.* The proof is a standard local finiteness argument in both the infinite and bi-infinite case. We give the bi-infinite case. Write $\alpha$ as the edge path $(\ldots, e_{-1}, e_0, e_1, \ldots)$ in $\Lambda$. Let $v_i$ be the initial point of $e_i$. By theorem 5.5, there is a $\Lambda$-geodesic $\beta_n$ which $K$-tracks $\alpha_n \equiv (e_{-n}, \ldots, e_n)$. Note that every vertex of $\beta_n$ is within $2K$ of a vertex of $\alpha$. For each positive integer $n$, some vertex $x_n$ of $\alpha_n$ is within $K$ of $v_0$. Hence there is an infinite number of $x_n$ that are equal. Of this infinite subcollection of $x_n$, infinitely many have the same pair of edges one preceding and one following $x_n$ on $\beta_n$, of this infinite collection of $x_n$ there is an infinite subcollection that have the same four edges - the two preceding and the two following $x_n$ being exactly the same. Continuing, we have a bi-infinite geodesic $\beta$ and each vertex of $\beta$ is within $2K$ of a vertex of $\alpha$. As $\alpha$ is a $(\lambda, \epsilon)$-quasi-geodesic, lemma 4.1 implies each point of $\alpha$ is within $\lambda(4K + 1) + \epsilon + 2K$ of $\beta$. $\square$

The next result follows directly from proposition 4.2 and corollary 6.1.

**Corollary 6.2.** *Suppose $(W, S)$ is a finitely generated Coxeter system, and $\Lambda(W, S)$ is the Cayley graph of $W$ with respect to $S$. Then a quasi-geodesic edge path ray in $\Lambda$ is tracked by a geodesic iff it has bounded bracket number.*

A metric space $(X, d)$ is a called a *geodesic metric space* if every pair of points are joined by a geodesic. It is *proper* if for any $x \in X$, the ball of radius $r$ about $X$ is compact for all positive numbers $r$. A group $W$ acts *geometrically* on a space if the action is properly discontinuous, co-compact and by isometries.

Let $(X, d)$ be a proper complete geodesic metric space. If $\triangle\ abc$ is a geodesic triangle in $X$, then we consider $\triangle\ \overline{a}\overline{b}\overline{c}$ in $\mathbb{E}^2$, a triangle with the same side lengths, and call this a *comparison triangle.* Then we say $X$ satisfies the $CAT(0)$ *inequality* if given $\triangle\ abc$ in $X$, then for any comparison triangle and any two points $p, q$ on $\triangle\ abc$, the corresponding points $\overline{p}, \overline{q}$ on the comparison triangle satisfy

$$d(p, q) \leq d(\overline{p}, \overline{q})$$

If $(X, d)$ is a $CAT(0)$ space, then the following basic properties hold:

(1) The distance function $d\colon X \times X \to \mathbb{R}$ is convex.
(2) $X$ has unique geodesic segments between points.
(3) $X$ is contractible.

For details, see [BH].

Suppose $(W, S)$ is a finitely generated Coxeter system, $\Lambda(W, S)$ is the Cayley graph of $W$ with respect to $S$ and $W$ acts geometrically on a CAT(0) space $X$. Define $\Lambda_x \subset X$ to have as vertices, the orbit $Wx$, and CAT(0) geodesic edge connecting $w_1 x$ and $w_2 x$ (for $w_i \in W$) when there is $s \in S$ such that $w_1 s = w_2$. There is a proper $W$-equivariant map $P_x : \Lambda \to \Lambda_x$ so that $P_x$ maps the identity vertex of $\Lambda$ to $x$.

Intuitively, the next result says that when a Coxeter group acts geometrically on a CAT(0) space, CAT(0) geodesics are tracked by Cayley graph geodesics. This result generalizes the right angled version of the same result in [MRT].

**Corollary 6.3.** *Suppose $(W, S)$ is a finitely generated Coxeter system, and $\Lambda(W, S)$ is the Cayley graph of $W$ with respect to $S$ and $W$ acts geometrically on the proper CAT(0) space $X$. If $x$ is a point of $X$ not fixed by any element of $W$, and $\Lambda_x$ is the copy of $\Lambda$ at $x$, then any CAT(0) geodesic ray in $X$ is tracked by a Cayley graph geodesic in $\Lambda_x$.*

*Proof.* For a given CAT(0) geodesic $\alpha$ we find a Cayley graph geodesic $\beta$ such that $P_x(\beta)$ tracks $\alpha$. It suffices to find $\lambda$, $\epsilon$, $K$ and $B$ such that any (finite) CAT(0) geodesic $\alpha$ is $K$-tracked by a Cayley $(\lambda, \epsilon)$-quasi-geodesic with bracket number $\leq B$. Since $W$ acts co-compactly on $X$, there is an integer $K_1$ such that every point of $X$ is within $K_1$ of the orbit $Wx$. For each integer $0, 1, \ldots, N$ such that $N$ is less that or equal to the length of $\alpha$, choose a point $v_i x$ of $Wx$ within $K_1$ of $\alpha(i)$. Let $\beta_i$ be a $\Lambda$-geodesic

connecting $v_i$ to $v_{i+1}$ and $\beta$ be the $\Lambda$-edge path $(\beta_0, \beta_1, \ldots)$. Since the map $P_x : \Lambda \to \Lambda_x$ is quasi-isometric, there are numbers $\lambda$ and $\epsilon$ such that any such $\beta$ is a $(\lambda, \epsilon)$-quasi-geodesic in $\Lambda$ and numbers $D_\Lambda$ and $D_X$ such that the length of any $\beta_i$ is less than or equal to $D_\Lambda$ (in $\Lambda$) and every point of such a $P_x(\beta_i)$ is within $D_X$ of $\alpha(i)$ (in $X$). Certainly every point of $\alpha$ is within $K \equiv K_1 + 1$ of $P_x(\beta)$.

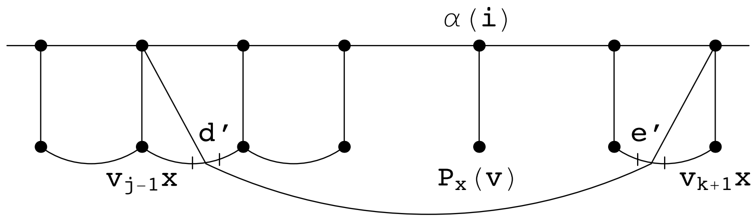


Figure 5

Hence it suffices to bound the bracket number of such a $\beta$. If $v$ is a vertex of $\beta_i$ and $e$ and $d$ are edges of $\beta$ preceding and following $v$ respectively such that $e$ and $d$ belong to the same wall $Q$ of $\Lambda$, then $e$ is an edge of $\beta_j$ and $d$ is an edge of $\beta_k$ where $j \leq i \leq k$. The mid-points $e'$ of $P_x(e)$ and $d'$ of $P_x(d)$ are fixed (in $\Lambda_x$ and $X$) by the reflection $r_Q \in W$ for the wall $Q$ (as are the mid-points of $e$ and $d$ in $\Lambda$). Hence the geodesic in $X$ connecting $d'$ and $e'$ is fixed by $r_Q$. Now, $d'$ (respectively $e'$) is within $D_X$ of $\alpha(j-1)$ (respectively $\alpha(k+1)$) and $P_x(v)$ is within $D_X$ of $\alpha(i)$. By the CAT(0) inequality for quadrilaterals (in particular for the quadrilateral determined by $d'$, $e'$, $\alpha(j-1)$, and $\alpha(k+1)$) $\alpha(i)$ is within $D_X$ of a point of the $X$-geodesic connecting $d'$ to $e'$ and hence $\alpha(i)$ is within $D_X$ of a fixed point of $r_Q$. (See figure 5.)

Since the action of $W$ on $X$ is properly discontinuous, there is a bound $B$ on the number of reflections $r_Q$ such that $r_Q$ does not take the ball of radius $D_X$ centered at $v(i) \in X$ (equivalently centered at any $x \in X$) off of itself. Hence there cannot be more than $B$ walls bracketing the vertex $v$ of $\beta$. $\square$

**Remark 6.4.** *Note that the above proof is valid even when $W$ does not act co-compactly on the CAT(0) space $X$, as long as the CAT(0) geodesic remains a bounded distance from $\Lambda_x$ for some $x$.*

The following result answers a question posed by K. Ruane.

**Corollary 6.5.** *Suppose $(W, S)$ is a finitely generated Coxeter group acting geometrically on the CAT(0) space $X$. For $x \in X$ let $\Lambda_x$ be the copy of the Cayley graph of $(W, S)$ in $X$, (with $W$-equivariant map $P_x : \Lambda(W, S) \to \Lambda_x$). Then for each subset $A \subset S$, the subgroup $\langle A \rangle$ is quasi-convex in $X$. (I.e. $P_x(\langle A \rangle)$ is quasi-convex in $X$.)*

*Proof.* Let $K$ be the tracking constant from corollary 6.3. Suppose $a_1, a_2 \in A$ and $\alpha$ is a CAT(0) geodesic in $X$ from $P_x(a_1)$ to $P_x(a_2)$. Let $\beta$ be a $\Lambda_x$,

edge path geodesic which $K$-tracks $\alpha$. I.e. there is a $\Lambda(W,S)$ geodesic $\beta'$, from $a_1$ to $a_2$ such that $P_x(\beta')=\beta$. Since $a_i \in A$, the edge labels of $\beta'$ are all in $A$. This means all vertices of $\beta'$ are in $\langle A\rangle$, and so the image of $\alpha$ is within $K$ of $P_x(\langle A\rangle)$. □

The next result says that elements of infinite order in a Coxeter group are tracked by geodesics in the standard Cayley graph.

**Corollary 6.6.** *Suppose $(W,S)$ is a finitely generated Coxeter system and $g \in W$ is an element of infinite order. Then in the Cayley graph $\Lambda(W,S)$ the elements $\{\ldots, g^{-2}, g^{-1}, 1, g, g^2, \ldots\}$ are tracked by a Cayley graph geodesic.*

*Proof.* By G. Moussong [Mo], all finitely generated Coxeter groups are CAT(0). Let $X$ be any CAT(0) space such that $W$ acts geometrically on $X$. The min set of $g$ contains a geodesic line $l$ that is invariant under the action of $g$. Let $x$ be any point in $X$ and $\Lambda_x$ the copy of $\Lambda(W,S)$ in $X$ at $x$. Let $\alpha$ be an $S$-geodesic for $g$. Observe that the edge path line $l_g$ in $\Lambda_x$ determined by positive and negative iterates of $\alpha$ at $x$ is a bounded distance from $l$. The proof of corollary 6.3 shows that $l_g$ is a quasi-geodesic with bounded bracket number and so by corollary 6.1 is tracked by a Cayley graph geodesic. □

One of the fundamental asymptotic results for word hyperbolic groups is that 1-ended word hyperbolic groups have locally connected boundary. This result follows from a long program of results by several authors, notably B. Bowditch, and concluded by G. Swarup [S]. To give a feeling for the reach of our results, we outline an elementary proof of this fact for Coxeter groups.

**Corollary 6.7.** *If $W$ is a 1-ended word hyperbolic Coxeter group then the boundary of $W$ is locally connected.*

*Proof.* We use an elementary form of a construction of a "filter" in [MRT] (where a partial classification of right angled Coxeter groups with locally connected boundaries is produced). Suppose $W$ acts geometrically on the CAT(0) space $X$, with base point $x$. Let $\Lambda_x$ be the copy of the Cayley graph of $(W,S)$ at $x$ in $X$ with proper $W$-equivariant map $P_x : \Lambda(W,S) \to \Lambda_x$. Suppose $r$ and $s$ are "close" geodesic rays in $X$, with $r(0) = s(0) = x$. Choose $\Lambda$ (edge path) geodesics $r'$ and $s'$ at $*$ (the identity vertex of $\Lambda(W,S)$), such that $P_x(r')$ and $P_x(s')$ $K$-track $r$ and $s$ respectively. Since $r$ and $s$ are close in $\partial X$, we may assume that $r'$ and $s'$ have long initial segments with "close" terminal points. For simplicity we assume these initial segments agree. If $y$ is the last vertex of this common initial segment, say the edge of $r'$ following $y$ has label $a_1$ and the edge of $s'$ following $y$ has lablel $b_1$. The presentation diagram $\Gamma(W,S)$ of $(W,S)$ has vertex set $S$ and an edge labeled $m(i,j)$ between distinct vertices $s_i, s_j$ if $m(i,j) \neq \infty$. Since $W$ is 1-ended no subset $A$ of $S$ with $\langle A\rangle$ a finite group separates $\Gamma$ (see corollary 16 of [MT]). The set $B$ of $S$-elements that label edges at $y$ with end points closer to $*$ than $y$ is to $*$ generates a finite subgroup of $W$. The set of vertices of $\Gamma$ corresponding to $B$ does not separate $\Gamma$ and $B$ does not

contain $a_1$ or $b_1$. Hence there is an edge path in $\Gamma$ from $a_1$ to $b_1$ avoiding $B$. Let the consecutive vertices of this path be $a_1 = v_1, v_2, \ldots, v_n = b_1$. If $q(i, i+1)$ is the (finite) order of $v_i v_{i+1}$ then the relation $(v_i, v_{i+1})^{q(i,i+1)}$ determines a loop at $y \in \Lambda$ such that the two half loops at $y$ making up this loop extend the Cayley geodesic from $*$ to $y$. Consider the subgraph $F_1$ of $\Lambda$ determined by the edge paths $r'$, $s'$ and the edge loops for each $v_i v_{i+1}$. Each $v_i$ determines an edge of $F_1$ (with label $v_i$) beginning at $y$. At the end point of this edge there are two edges of $F_1$ that extend a Cayley geodesic from $*$ to $y$. Build a set of loops as with $a_1$ and $b_1$ for each of these pairs of edges. Then $F_2$ is $F_1$ union all new loops. Continuing we build a 1-ended subgraph $F = \cup_{i=1}^{\infty} F_i$ of $\Lambda$ such that for each vertex $v$ of $F$, not on the common overlap of $r'$ and $s'$, there is a Cayley geodesic from $*$ to $v$ in $F$ which passes through $y$. We claim that $L$, the limit set of $P_x(F)$ is a "small" connected set containing $r$ and $s$ (and so $\partial X$ is locally connected). Certainly, $r$ and $s$ are in $L$. Since $F$ is 1-ended and $P_x$ is proper, $L$ is connected. If $v$ is a vertex of $F$, then there is a Cayley geodesic $\alpha_v$ from $*$ to $v$ (which passes through $y$ for all but finitely many $v$). If $z \in L$ then let $z_1, z_2, \ldots$ be a sequence of vertices of $F$ such that $P_x(z_i)$ converges to $z$. The CAT(0) geodesic from $x$ to $P_x(z_i)$ is $K$- tracked by a Cayley geodesic $\beta_i$ in $\Lambda_x$. As $W$ is word hyperbolic the Cayley geodesics $P_x(\alpha_{v_i})$ and $\beta_i$ (with the same end points) must $\delta$-fellow travel (for a fixed constant $\delta$). In particular each $\beta_i$ must pass "close" to $P_x(y)$ and so $z$ is close to both $r$ and $s$ in $\partial X \equiv \partial W$. $\square$

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